

ON FRACTIONAL POINCARÉ INEQUALITIES

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ABSTRACT. We show that fractional (p, p) -Poincaré inequalities and even fractional Sobolev–Poincaré inequalities hold for bounded John domains, and especially for bounded Lipschitz domains. We also prove sharp fractional $(1, p)$ -Poincaré inequalities for s -John domains.

1. INTRODUCTION

We consider the following fractional (q, p) -Poincaré inequality in a bounded domain G in \mathbb{R}^n , $n \geq 2$,

$$(1.1) \quad \int_G |u(x) - u_G|^q dx \leq c \left(\int_G \int_{G \cap B^n(x, \tau \operatorname{dist}(x, \partial G))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p},$$

where $1 \leq p, q < \infty$, $\delta, \tau \in (0, 1)$, and the constant c does not depend on $u \in L^p(G)$. Our inequality (1.1) with $q = p$ is stronger than the fractional inequality

$$(1.2) \quad \int_G |u(x) - u_G|^p dx \leq c \int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dx dy,$$

where on the right hand side is the commonly used seminorm on $W^{\delta, p}(G)$, [A]. Augusto C. Ponce showed that bounded Lipschitz domains support the same type of inequalities as (1.2) but with general radial weights, [P1], [P2, Theorem 1.1]. Jean Bourgain, Haïm Brezis, and Petru Mironescu found the optimal constant c in (1.2) when G is a cube [BBM2, Theorem 1]. An elementary proof was provided by Vladimir Maz'ya and Tatyana Shaposhnikova, [MS1], [MS2]. The relationship between the right hand side of (1.2) and the $L^p(G)$ integrability of the absolute value of the gradient in smooth bounded domains is considered in [BBM1].

We give sufficient geometric conditions for a bounded domain G in \mathbb{R}^n to support the fractional (q, p) -Poincaré inequality for $1 \leq q \leq p < \infty$, Theorem 3.1. Examples of the domains which support the fractional (p, p) -Poincaré inequality are John domains, Theorem 4.3. The John domains include uniform domains and hence also Lipschitz

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domains. We show that John domains support the fractional Poincaré inequality (1.1) when $1 < p \leq q \leq np/(n - \delta p)$ and $p < n/\delta$, Theorem 4.10. We also study more general bounded domains, so called s -John domains with $s > 1$. We prove fractional $(1, p)$ -Poincaré inequalities for these domains, Theorem 5.1, and we show that these results are sharp, Theorem 6.9.

2. NOTATION AND AUXILIARY RESULTS

We assume that G is a bounded domain in Euclidean n -space \mathbb{R}^n , $n \geq 2$, throughout the paper.

We denote by \mathcal{D} the family of closed dyadic cubes in \mathbb{R}^n . We let \mathcal{D}_j be the family of those dyadic cubes whose side length is 2^{-j} , $j \in \mathbb{Z}$. For a domain G we fix its Whitney decomposition $W = W_G \subset \mathcal{D}$. For the properties of dyadic cubes and Whitney cubes we refer to Elias M. Stein's book, [S]. We write $Q^* = \frac{9}{8}Q$ for $Q \in W$. Then,

$$(2.1) \quad \frac{3}{4} \text{diam}(Q) \leq \text{dist}(x, \partial G) \leq 6 \text{diam}(Q), \quad \text{if } x \in Q^*.$$

Let us fix a cube Q_0 in the Whitney decomposition W . For each $Q \in W$ there exists a chain of cubes $(Q_0^*, Q_1^*, \dots, Q_k^*) =: C(Q^*)$ joining two cubes Q_0^* and $Q_k^* = Q^*$ such that $Q_i^* \cap Q_j^* \neq \emptyset$ if and only if $|i - j| \leq 1$. The length of this chain is written as $\ell(C(Q^*)) := k$. Once the chains of cubes have been picked up, then for each Whitney cube A we define a set $A(W) = \{Q \in W \mid A^* \in C(Q^*)\}$. We call this construction a chain decomposition of G with a fixed cube Q_0 .

The side length of a cube Q in \mathbb{R}^n is denoted by $\ell(Q)$. We write χ_E for the characteristic function of a set E . The Lebesgue n -measure of a measurable set E is denoted by $|E|$. The upper Minkowski dimension of a set E in \mathbb{R}^n is

$$\dim_{\mathcal{M}}(E) := \sup \{ \lambda \geq 0 \mid \limsup_{r \rightarrow 0+} \mathcal{M}_\lambda(E, r) = \infty \},$$

where

$$\mathcal{M}_\lambda(E, r) := \frac{|E + B^n(0, r)|}{r^{n-\lambda}} = \frac{|\cup_{x \in E} B^n(x, r)|}{r^{n-\lambda}}, \quad r > 0,$$

is the λ -dimensional Minkowski precontent.

The notation $a \lesssim b$ is used to express that an estimate $a \leq cb$ holds for some constant $c > 0$ whose value is clear from the context. We use subscripts to indicate the dependence on parameters, for example, a quantity c_λ depends on a parameter λ .

The following lemma gives a fractional inequality in a cube.

2.2. Lemma. *Let Q be a closed cube in \mathbb{R}^n . Let $1 \leq q \leq p < \infty$ and let $\delta, \rho \in (0, 1)$. Then, there is a constant $c < \infty$ independent of $u \in L^p(Q)$*

such that

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |u(y) - u_Q|^q dy \\ & \leq c|Q|^{q(\delta/n-1/p)} \left(\int_Q \int_{Q \cap B^n(y, \rho \ell(Q))} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}. \end{aligned}$$

Proof. Without loss of generality we may assume that $Q = [0, 1]^n$. This comes from a simple scaling and translation argument.

Let us divide Q into k^n congruent and closed subcubes Q_1, \dots, Q_{k^n} , where k is chosen such that $R \subset B^n(y, \rho)$ for every $y \in R$ whenever R is a union of two cubes Q_i and Q_j , $i, j \in \{1, 2, \dots, k^n\}$, sharing a common face; in particular, the case $i = j$ is allowed. We obtain

$$\begin{aligned} (2.3) \quad & \frac{1}{|R|} \int_R |u(y) - u_R|^q dy \leq \left(\frac{1}{|R|} \int_R |u(y) - u_R|^p dy \right)^{q/p} \\ & \leq \left(\frac{1}{|R|} \int_R \frac{1}{|R|} \int_R |u(y) - u(z)|^p dz dy \right)^{q/p} \\ & \lesssim |R|^{q(\delta/n-1/p)} \left(\int_R \int_R \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p} \\ & \lesssim \left(\int_Q \int_{Q \cap B^n(y, \rho)} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}. \end{aligned}$$

Hölder's inequality and Minkowski's inequality yield

$$\begin{aligned} (2.4) \quad & \frac{1}{|Q|} \int_Q |u(y) - u_Q|^q dy \lesssim \frac{1}{|Q|} \int_Q |u(y) - u_{Q_1}|^q dy \\ & \lesssim \sum_{j=1}^{k^n} \int_{Q_j} |u(y) - u_{Q_j}|^q dy + \sum_{j=1}^{k^n} \int_{Q_j} |u_{Q_j} - u_{Q_1}|^q dy. \end{aligned}$$

By (2.3) it is enough to estimate the second series in (2.4). Let us fix Q_j , $j \in \{1, \dots, k^n\}$, and let $\sigma : \{1, 2, \dots, kn\} \rightarrow \{1, 2, \dots, k^n\}$ be such that $\sigma(1) = 1$, $\sigma(kn) = j$, and the subsequent cubes $Q_{\sigma(i)}$ and $Q_{\sigma(i+1)}$ share a common face if $i = 1, \dots, kn - 1$. Since $kn \lesssim 1$, we obtain

$$\begin{aligned} (2.5) \quad & |u_{Q_j} - u_{Q_1}|^q \leq \left(\sum_{i=1}^{kn-1} |u_{Q_{\sigma(i+1)}} - u_{Q_{\sigma(i)}}| \right)^q \\ & \lesssim \sum_{i=1}^{kn-1} |u_{Q_{\sigma(i+1)}} - u_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q + \sum_{i=1}^{kn-1} |u_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}} - u_{Q_{\sigma(i)}}|^q. \end{aligned}$$

Let us consider the first sum in (2.5). Note that

$$\begin{aligned}
& |u_{Q_{\sigma(i+1)}} - u_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q \\
& \leq \frac{1}{|Q_{\sigma(i+1)}|} \int_{Q_{\sigma(i+1)}} |u_{Q_{\sigma(i+1)}} - u(y) + u(y) - u_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q dy \\
& \lesssim \frac{1}{|Q_{\sigma(i+1)}|} \int_{Q_{\sigma(i+1)}} |u(y) - u_{Q_{\sigma(i+1)}}|^q dy \\
& \quad + \frac{1}{|Q_{\sigma(i+1)} \cup Q_{\sigma(i)}|} \int_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}} |u(y) - u_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q dy.
\end{aligned}$$

By (2.3) we obtain

$$\sum_{i=1}^{kn-1} |u_{Q_{\sigma(i+1)}} - u_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q \lesssim \left(\int_Q \int_{Q \cap B^n(y, \rho)} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}.$$

Similar estimates for the remaining sum in (2.5) conclude the proof. \square

We also need some estimates involving porous sets in \mathbb{R}^n .

2.6. Definition. A set S in Euclidean n -space is *porous in \mathbb{R}^n* if for some $\kappa \in (0, 1]$ the following statement is true: for every $x \in \mathbb{R}^n$ and $0 < r \leq 1$ there is $y \in B^n(x, r)$ such that $B^n(y, \kappa r) \cap S = \emptyset$.

The following lemma gives a norm estimate related to porous sets, and it is based on maximal function techniques. This lemma might be of independent interest.

2.7. Lemma. *Suppose that S is porous in \mathbb{R}^n and let $1 \leq p < \infty$. If $x \in S$ and $0 < r \leq 1$, then*

$$\int_{B^n(x, r)} \log^p \frac{1}{\text{dist}(y, S)} dy \leq cr^n(1 + \log^p r^{-1}),$$

where the constant c is independent of x and r .

Proof. Let us write

$$C_S = \{R \in \mathcal{D} : \text{dist}(x_R, S)/(4 + \sqrt{n}) \leq \ell(R) \leq 1\},$$

where x_R is the midpoint of a dyadic cube R . Suppose that $R \in \mathcal{D}$ is such that $\ell(R) \leq 1$ and $\text{dist}(y, S) \leq 4\ell(R)$ for some $y \in R$. Then, since

$$\begin{aligned}
(2.8) \quad \text{dist}(x_R, S) & \leq \text{dist}(x_R, y) + \text{dist}(y, S) \\
& \leq \sqrt{n}\ell(R) + \text{dist}(y, S) \leq (4 + \sqrt{n})\ell(R)
\end{aligned}$$

for the midpoint of R , we conclude that $R \in C_S$.

Fix $j \in \mathbb{N}_0$ such that $2^{-j} \leq r < 2^{-j+1}$, and consider a dyadic cube $Q \in \mathcal{D}_j$ for which $Q \cap B^n(x, r) \neq \emptyset$. By covering $B^n(x, r)$ with such dyadic cubes it is enough to show that

$$(2.9) \quad \|\log \text{dist}(\cdot, S)^{-1}\|_{L^p(Q \cap B^n(x, r))}^p \lesssim r^n(1 + \log^p r^{-1}).$$

By the porosity and the Lebesgue density theorem, the n -measure of S is zero. Hence, it is enough to consider points $y \in Q \cap B^n(x, r) \setminus S$. Since $x \in S$,

$$(2.10) \quad 1 \leq \frac{2\ell(Q)}{\text{dist}(y, S)}.$$

Let us consider a finite sequence of dyadic cubes

$$Q = Q_0(y) \supset Q_1(y) \supset \cdots \supset Q_m(y),$$

each of them containing the point y and satisfying

$$(2.11) \quad \ell(Q_i(y))/\ell(Q_{i+1}(y)) = 2, \quad i = 0, 1, \dots, m-1.$$

The last cube is chosen to satisfy

$$(2.12) \quad \text{dist}(y, S)/4 \leq \ell(Q_m(y)) < \text{dist}(y, S)/2.$$

From (2.10) it follows that $m \geq 1$. By (2.11) and (2.10)

$$2^m = \prod_{i=0}^{m-1} \frac{\ell(Q_i(y))}{\ell(Q_{i+1}(y))} = \frac{\ell(Q_0(y))}{\ell(Q_m(y))} > \frac{2\ell(Q_0(y))}{\text{dist}(y, S)} = \frac{2\ell(Q)}{\text{dist}(y, S)} \geq 1.$$

Hence,

$$m \geq \log 2^m \geq \log 2\ell(Q) - \log \text{dist}(y, S) \geq 0.$$

Furthermore, (2.12) and (2.8) yield $Q_i(y) \in C_S$ if $i = 0, 1, \dots, m$. Thus, we obtain

$$\sum_{\substack{R \in C_S \\ R \subset Q}} \chi_R(y) \geq 1 + m \geq 1 + \log(\ell(Q)) - \log \text{dist}(y, S) \geq 0,$$

where χ_R is the characteristic function of R . Integrating this inequality and using triangle-inequality yields

$$\begin{aligned} & \|\log \text{dist}(\cdot, S)^{-1}\|_{L^p(Q \cap B^n(x, r))} \\ & \leq |1 + \log \ell(Q)| |Q \cap B^n(x, r)|^{1/p} + \left\| \sum_{\substack{R \in C_S \\ R \subset Q}} \chi_R \right\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Since S is porous in \mathbb{R}^n , we may follow the proof of [IV, Theorem 2.10]. We obtain a constant K_κ , depending on κ in Definition 2.6, and families

$$\{\hat{R}\}_{R \in C_S^k}, \quad C_S^k \subset C_S, \quad k = 0, 1, \dots, K_\kappa - 1,$$

where each $\{\hat{R}\}_{R \in C_S^k}$ is a disjoint family of cubes $\hat{R} \subset R$, such that

$$\left\| \sum_{\substack{R \in C_S \\ R \subset Q}} \chi_R \right\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{k=0}^{K_\kappa-1} \left\| \sum_{\substack{R \in C_S^k \\ R \subset Q}} \chi_{\hat{R}} \right\|_{L^p(\mathbb{R}^n)} \leq \sum_{k=0}^{K_\kappa-1} \|\chi_Q\|_{L^p(\mathbb{R}^n)} \lesssim |Q|^{1/p}.$$

By combining the estimates we obtain

$$\|\log \text{dist}(\cdot, S)^{-1}\|_{L^p(Q \cap B^n(x, r))} \lesssim (1 + \log \ell(Q)^{-1}) |Q|^{1/p} \lesssim (1 + \log r^{-1}) r^{n/p}.$$

Estimate (2.9) follows. \square

3. CONDITIONS FOR THE FRACTIONAL POINCARÉ INEQUALITY

In the following theorem we give sufficient conditions for a bounded domain to support the fractional (q, p) -Poincaré inequality (1.1).

3.1. Theorem. *Let G be a bounded domain in n -dimensional Euclidean space, $n \geq 2$, with a Whitney decomposition W . Let $1 \leq q \leq p < \infty$ and let $\delta, \tau \in (0, 1)$.*

(1) If $q < p$ and if there exists a chain decomposition of G such that

$$(3.2) \quad \sum_{A \in W} \left(\sum_{Q \in A(W)} \ell(C(Q^*))^{q-1} |Q| |A|^{q(\delta/n-1/p)} \right)^{p/(p-q)} < \infty,$$

then G supports the fractional (q, p) -Poincaré inequality (1.1).

(2) If $q = p$ and if there exists a chain decomposition of G such that

$$(3.3) \quad \sup_{A \in W} \sum_{Q \in A(W)} \ell(C(Q^*))^{p-1} |Q| |A|^{p\delta/n-1} < \infty,$$

then G supports the fractional (p, p) -Poincaré inequality (1.1).

Proof. We prove (1); the proof of (2) is similar. Let δ and τ in $(0, 1)$ be given. We use Hölder's inequality and Minkowski's inequality and then the Whitney decomposition to obtain

$$(3.4) \quad \begin{aligned} \int_G |u(x) - u_G|^q dx &\lesssim \int_G |u(x) - u_{Q_0^*}|^q dx \\ &\leq \sum_{Q \in W} \int_{Q^*} |u(x) - u_{Q_0^*}|^q dx \\ &\lesssim \sum_{Q \in W} \int_{Q^*} |u(x) - u_{Q^*}|^q dx + \sum_{Q \in W} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q dx. \end{aligned}$$

Lemma 2.2 with $\rho = 2\tau/3$ yields

$$\begin{aligned} &\int_{Q^*} |u(x) - u_{Q^*}|^q dx \\ &\lesssim |Q^*|^{1+q(\delta/n-1/p)} \left(\int_{Q^*} \int_{Q^* \cap B^n(y, \rho \ell(Q^*))} \frac{|u(z) - u(y)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}. \end{aligned}$$

Inequalities (2.1) and $(1 + q\delta/n - q/p)(p/(p - q)) > 1$ imply

$$\begin{aligned}
& \sum_{Q \in W} \int_{Q^*} |u(x) - u_{Q^*}|^q dx \\
& \lesssim \sum_{Q \in W} |Q|^{1+q\delta/n-q/p} \left(\int_{Q^*} \int_{Q^* \cap B^n(y, \rho\ell(Q^*))} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p} \\
& \leq \left(\sum_{Q \in W} (|Q|^{1+q\delta/n-q/p})^{p/(p-q)} \right)^{(p-q)/p} \\
& \quad \left(\sum_{Q \in W} \int_{Q^*} \int_{Q^* \cap B^n(y, \rho\ell(Q^*))} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p} \\
& \lesssim \left(\int_G \int_{G \cap B^n(y, \tau \text{dist}(y, \partial G))} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}.
\end{aligned}$$

Next, we estimate the latter sum in (3.4). By using chains from the chain decomposition we obtain

$$\begin{aligned}
\sum_{Q \in W} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q dx & \lesssim \sum_{Q \in W} |Q| \left(\sum_{j=1}^k |u_{Q_j^*} - u_{Q_{j-1}^*}| \right)^q \\
& \leq \sum_{Q \in W} \ell(C(Q^*))^{q-1} |Q| \left(\sum_{j=1}^k |u_{Q_j^*} - u_{Q_{j-1}^*}|^q \right).
\end{aligned}$$

Estimate $\max\{|Q_j^*|, |Q_{j-1}^*|\} \lesssim |Q_j^* \cap Q_{j-1}^*|$ and Hölder's inequality yield

$$\begin{aligned}
|u_{Q_j^*} - u_{Q_{j-1}^*}|^q & \lesssim \sum_{i=j-1}^j \left(|Q_i^*|^{-1} \int_{Q_i^*} |u(x) - u_{Q_i^*}| dx \right)^q \\
& \leq \sum_{i=j-1}^j |Q_i^*|^{-1} \int_{Q_i^*} |u(x) - u_{Q_i^*}|^q dx.
\end{aligned}$$

Lemma 2.2 with $\rho = 2\tau/3$ implies

$$\begin{aligned}
& |u_{Q_j^*} - u_{Q_{j-1}^*}|^q \\
& \lesssim \sum_{i=j-1}^j |Q_i^*|^{q(\delta/n-1/p)} \left(\int_{Q_i^*} \int_{Q_i^* \cap B^n(y, \rho\ell(Q_i^*))} \frac{|u(z) - u(y)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}.
\end{aligned}$$

We have obtained for the second sum in (3.4)

$$\begin{aligned}
& \sum_{Q \in W} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q dx \\
& \lesssim \sum_{Q \in W} \ell(C(Q^*))^{q-1} |Q| \\
& \quad \left(\sum_{j=0}^k |Q_j^*|^{q(\delta/n-1/p)} \left(\int_{Q_j^*} \int_{Q_j^* \cap B^n(y, \rho\ell(Q_j^*))} \frac{|u(z) - u(y)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p} \right).
\end{aligned}$$

When we rearrange the double sum, we obtain

$$\begin{aligned} & \sum_{Q \in W} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q dx \\ & \lesssim \sum_{A \in W} \sum_{Q \in A(W)} \ell(C(Q^*))^{q-1} |Q| |A|^{q(\delta/n-1/p)} \\ & \quad \left(\int_{A^*} \int_{A^* \cap B^n(y, \rho \ell(A^*))} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}. \end{aligned}$$

Hölder's inequality with $(\frac{p}{q}, \frac{p}{p-q})$, and inequalities (3.2) and (2.1) yield

$$\begin{aligned} \sum_{Q \in W} \int_{Q^*} |u_{Q^*} - u_{Q_0^*}|^q dx & \lesssim \left(\sum_{A \in W} \int_{A^*} \int_{A^* \cap B^n(y, \rho \ell(A^*))} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p} \\ & \lesssim \left(\int_G \int_{G \cap B^n(y, \tau \text{dist}(y, \partial G))} \frac{|u(y) - u(z)|^p}{|z - y|^{n+\delta p}} dz dy \right)^{q/p}. \end{aligned}$$

Hence, G supports the fractional (q, p) -Poincaré inequality (1.1). \square

3.5. *Remark.* Let G be a bounded domain in \mathbb{R}^n and let $1 \leq p < \infty$. By [Hu, Theorem 6.6] the estimate

$$(3.6) \quad \sup_{A \in W} \sum_{Q \in A(W)} \ell(C(Q^*))^{p-1} |Q| |A|^{p/n-1} < \infty$$

is a sufficient condition for the classical (p, p) -Poincaré inequality to be valid in the domain G . A comparison to our sufficient condition (3.3) for the *fractional* (p, p) -Poincaré inequality shows that condition (3.6) for the classical (p, p) -Poincaré inequality is weaker.

4. POSITIVE RESULTS FOR 1-JOHN DOMAINS

As an application of Theorem 3.1 we show that 1-John domains support the fractional (p, p) -Poincaré inequality, Theorem 4.3. We also consider fractional Sobolev–Poincaré inequalities, Theorem 4.10 and Remark 4.14. We recall that bounded uniform and Lipschitz domains are examples of 1-John domains.

4.1. **Definition.** A bounded domain G in \mathbb{R}^n , $n \geq 2$, is an *s-John domain*, $s \geq 1$, if there is a point x_0 in G and a constant $c > 0$ such that every point x in G can be joined to x_0 by a rectifiable curve $\gamma : [0, l] \rightarrow G$ parametrized by its arc length for which $\gamma(0) = x$, $\gamma(l) = x_0$, $l \leq c$, and

$$\text{dist}(\gamma(t), \partial G) \geq t^s/c \quad \text{for } t \in [0, l].$$

The point x_0 is called an *s-John center* of G .

If G is a 1-John domain, then its boundary ∂G is porous in \mathbb{R}^n , Definition 2.6. The boundary of an *s-John* domain with $s > 1$ may have positive Lebesgue n -measure, [N], and thus it is not necessarily porous in \mathbb{R}^n .

Let us construct a chain decomposition of a given s -John domain G . Let $Q \in W = W_G$ and fix a rectifiable curve γ that is parametrized by its arc length and joins the midpoints x_Q and $x_0 := x_{Q_0}$, Definition 4.1. Assume that x_{Q_0} lies in one of the cubes intersecting Q . Join x_Q to x_{Q_0} by an arc that is contained in $Q \cup Q_0$ and whose length is comparable to $\ell(Q)$. Otherwise there is $r > 0$ such that $\gamma(r)$ lies in the boundary of a Whitney cube P that intersects Q and $\gamma(t)$ belongs to a cube that is not intersecting Q whenever $t \in (r, \ell(\gamma)]$. Join the midpoint x_Q to the midpoint x_P by an arc whose length is comparable to $\ell(Q)$ and is in $Q \cup P$. We iterate these steps with Q replaced by P , and we continue until we reach x_{Q_0} . Let γ_Q be this composed curve parametrized by its arc length. It is straightforward to verify that $\ell(\gamma_Q) \leq c$ and

$$(4.2) \quad t^s/c \leq \text{dist}(\gamma_Q(t), \partial G) \quad \text{if } t \in [0, \ell(\gamma_Q)],$$

where $c > 0$ depends on the s -John constant of G , s , and n . Let $C(Q^*)$ be a chain consisting of cubes A^* such that $A \in W$ and $x_A \in \gamma_Q[0, \ell(\gamma_Q)]$.

For 1-John domains we first have the following result.

4.3. Theorem. *A 1-John domain G in \mathbb{R}^n supports the fractional (p, p) -Poincaré inequality (1.1) if $1 \leq p < \infty$ and $\tau, \delta \in (0, 1)$.*

Proof. We may assume that $\text{diam}(G) \leq 1$. By (4.2) with $s = 1$ and the fact that γ_Q , $Q \in W$, connects the midpoints of cubes in $C(Q^*)$,

$$(4.4) \quad \ell(C(Q^*)) \leq c \left(1 + \log \frac{1}{\ell(Q)}\right),$$

where the constant c is independent of Q . If $A \in W$, then

$$(4.5) \quad \bigcup_{Q \in A(W)} Q \subset B^n(\omega_A, \min\{1, c\ell(A)\}),$$

where ω_A is the closest point in ∂G to x_A and the constant $c > 0$ is independent of A . By (4.4) and (4.5) we obtain

$$\begin{aligned} \sum_{Q \in A(W)} \ell(C(Q^*))^{p-1} |Q| &\lesssim \sum_{Q \in A(W)} |Q| \left(1 + \log \frac{1}{\ell(Q)}\right)^{p-1} \\ &\lesssim \sum_{Q \in A(W)} |Q| \left(1 + \log^p \frac{1}{\ell(Q)}\right) \lesssim \sum_{Q \in A(W)} \int_Q \left(1 + \log^p \frac{1}{\text{dist}(y, \partial G)}\right) dy \\ &\leq \int_{B^n(\omega_A, \min\{1, c\ell(A)\})} \left(1 + \log^p \frac{1}{\text{dist}(y, \partial G)}\right) dy. \end{aligned}$$

Since ∂G is porous in \mathbb{R}^n , Lemma 2.7 yields

$$\sum_{Q \in A(W)} \ell(C(Q^*))^{p-1} |Q| \lesssim |A| (1 + \log^p \ell(A)^{-1}) \lesssim |A|^{1-\delta p/n}.$$

We have verified condition (3.3) in Theorem 3.1. Hence, the domain G supports the fractional (p, p) -Poincaré inequality. \square

We state an immediate corollary of Theorem 4.3.

4.6. Corollary. *Let G be a bounded domain in \mathbb{R}^n , $n \geq 2$, and let $1 \leq p < \infty$, $\delta, \tau \in (0, 1)$. Then G supports the fractional (p, p) -Poincaré inequality (1.1) if G is a uniform domain or a Lipschitz domain.*

It is well known [B, Theorem 5.1, Lemma 3.1] that 1-John domains support Sobolev–Poincaré inequalities: if $1 \leq p \leq q \leq np/(n-p)$, $p < n$, then there is $c > 0$ such that, for every $u \in W^{1,p}(G)$,

$$(4.7) \quad \left(\int_G |u(x) - u_G|^q dx \right)^{1/q} \leq c \left(\int_G |\nabla u(x)|^p dx \right)^{1/p}.$$

We consider the corresponding fractional Sobolev–Poincaré inequalities on 1-John domains, Theorem 4.10. For the proof of this theorem we need the Riesz potentials I_δ , $\delta \in (0, n)$, that are defined for suitable f by

$$I_\delta(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\delta}} dy.$$

A proof of the following theorem is in [He, Theorem 1].

4.8. Theorem. *Let $0 < \delta < n$, $1 < p < q < \infty$, and $1/p - 1/q = \delta/n$. Then $\|I_\delta(f)\|_q \leq c\|f\|_p$ for a constant $c > 0$ independent of $f \in L^p(\mathbb{R}^n)$.*

We also need the following chaining lemma. It is a slight modification of [HK, Theorem 9.3]: we add the new condition 3 but the proof adapts to our setting, and we omit the details.

4.9. Lemma. *Let G in \mathbb{R}^n be a 1-John domain whose 1-John constant is $c_J > 1$. Fix a number $M > 1$. Denote by $x_0 \in G$ the 1-John center of G , and let*

$$B_0 := B(x_0, \text{dist}(x_0, \partial G)/4Mc_J).$$

Then, there is a constant $c > 0$, depending on G , M , and n , as follows: given $x \in G$ there is a sequence of balls $B_i = B(x_i, r_i) \subset G$, $i = 0, 1, \dots$, such that for all $i = 0, 1, \dots$, the following conditions 1–5 hold:

1. $|B_i \cup B_{i+1}| \leq c|B_i \cap B_{i+1}|$;
2. $\text{dist}(x, B_i) \leq cr_i$;
3. $\text{dist}(B_i, \partial G) \geq Mr_i$;
4. $|x - x_i| \leq cr_i$ and $r_j \rightarrow 0$ as $j \rightarrow \infty$;
5. $\sum_{j=0}^{\infty} \chi_{B_j} \leq c\chi_G$.

The following result is a fractional Sobolev–Poincaré inequality for 1-John domains.

4.10. Theorem. *Assume that G is a 1-John domain in \mathbb{R}^n , $n \geq 2$. Suppose that $\tau, \delta \in (0, 1)$, $p < n/\delta$, and*

$$1 < p \leq q \leq \frac{np}{n - \delta p}.$$

Then G supports the fractional (q, p) -Poincaré inequality (1.1).

Proof. By Hölder's inequality we may assume that $q = np/(n - \delta p)$. Fix $\tau \in (0, 1)$ and let $u \in L^p(G)$. Let $x \in G$ be a Lebesgue point of u , and consider the associated balls $B_i = B(x_i, r_i)$ from Lemma 4.9 satisfying conditions 1–5 with $M > 2/\tau$.

The following holds: for all i ,

$$(4.11) \quad B_i \subset B^n(y, \tau \operatorname{dist}(y, \partial G)), \quad \text{if } y \in B_i.$$

Namely, let us fix $y \in B_i$ and let z be any point in B_i . Then, by condition 3 in Lemma 4.9,

$$\begin{aligned} |z - y| &\leq |y - x_i| + |x_i - z| \leq 2r_i \leq 2 \frac{\operatorname{dist}(B_i, \partial G)}{M} \\ &\leq \frac{2}{M} \operatorname{dist}(y, \partial G) < \tau \operatorname{dist}(y, \partial G). \end{aligned}$$

By the Lebesgue differentiation theorem and condition 4 in Lemma 4.9,

$$u(x) = \lim_{i \rightarrow \infty} \frac{1}{|B_i|} \int_{B_i} u(y) dy = \lim_{i \rightarrow \infty} u_{B_i}.$$

Hence, by condition 1 in Lemma 4.9, we obtain

$$\begin{aligned} |u(x) - u_{B_0}| &\leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\ &\leq \sum_{i=0}^{\infty} (|u_{B_i} - u_{B_i \cap B_{i+1}}| + |u_{B_{i+1}} - u_{B_i \cap B_{i+1}}|) \\ &\lesssim \sum_{i=0}^{\infty} \frac{1}{|B_i|} \int_{B_i} |u(y) - u_{B_i}| dy. \end{aligned}$$

For a ball B_i ,

$$\begin{aligned} \frac{1}{|B_i|} \int_{B_i} |u(y) - u_{B_i}| dy &= \frac{1}{|B_i|} \int_{B_i} \left| \frac{1}{|B_i|} \int_{B_i} (u(y) - u(z)) dz \right| dy \\ &\leq \frac{1}{|B_i|} \int_{B_i} \left(\frac{1}{|B_i|} \int_{B_i} |u(y) - u(z)|^p dz \right)^{1/p} dy \\ (4.12) \quad &= \frac{1}{|B_i|^{1+1/p}} \int_{B_i} \left(\int_{B_i} |u(y) - u(z)|^p dz \right)^{1/p} dy \\ &\lesssim |B_i|^{\delta/n-1} \int_{B_i} \left(\int_{B_i} \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz \right)^{1/p} dy. \end{aligned}$$

Let us write

$$g(y) := \left(\int_{G \cap B^n(y, \tau \operatorname{dist}(y, \partial G))} \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz \right)^{1/p}.$$

By (4.12), (4.11) and condition 2 in Lemma 4.9,

$$\begin{aligned}
\sum_{i=0}^{\infty} \frac{1}{|B_i|} \int_{B_i} |u(y) - u_{B_i}| dy &\lesssim \sum_{i=0}^{\infty} |B_i|^{\delta/n-1} \int_{B_i} \left(\int_{B_i} \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz \right)^{1/p} dy \\
&\leq \sum_{i=0}^{\infty} |B_i|^{\delta/n-1} \int_{B_i} \left(\int_{B^n(y, \tau \operatorname{dist}(y, \partial G))} \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz \right)^{1/p} dy \\
&\lesssim \sum_{i=0}^{\infty} r_i^{n(\delta/n-1)} \int_{B_i} g(y) dy \\
&\lesssim \sum_{i=0}^{\infty} \int_{B_i} \frac{g(y)}{|x - y|^{n-\delta}} dy.
\end{aligned}$$

By condition 5 in Lemma 4.9,

$$(4.13) \quad |u(x) - u_{B_0}| \lesssim \int_G \frac{g(y)}{|x - y|^{n-\delta}} dy = I_{\delta}(\chi_G g)(x)$$

for every Lebesgue point $x \in G$. By integrating this inequality and using Theorem 4.8, we obtain

$$\begin{aligned}
\left(\int_G |u(x) - u_{B_0}|^q dx \right)^{1/q} &\lesssim \|I_{\delta}(\chi_G g)\|_q \lesssim \|\chi_G g\|_p \\
&= \left(\int_G \int_{G \cap B^n(y, \tau \operatorname{dist}(y, \partial G))} \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz dy \right)^{1/p}.
\end{aligned}$$

Inequality (1.1) follows. \square

4.14. *Remark.* The proof of Theorem 4.10 also gives the following result: Suppose that G is a 1-John domain in \mathbb{R}^n . Let $\tau, \delta \in (0, 1)$ and let $p, q \in [1, \infty)$ be such that

$$0 \leq 1/p - 1/q < \delta/n.$$

Then G supports the fractional (q, p) -Poincaré inequality (1.1). Indeed, it suffices to recall that the linear operator $f \mapsto I_{\delta}(\chi_G f)$ is bounded from $L^p(G)$ to $L^q(G)$, [GT, Lemma 7.12].

5. POSITIVE RESULTS FOR s -JOHN DOMAINS WITH $s > 1$

We prove the fractional $(1, p)$ -Poincaré inequality (1.1) for s -John domains, Theorem 5.1. We show in Section 6 that this result is sharp in terms of the restriction on p , Theorem 6.9.

5.1. Theorem. *Let $s > 1$, $1 < p < \infty$, $\lambda \in [n - 1, n)$, and let $\delta, \tau \in (0, 1)$. Suppose that*

$$(5.2) \quad s < \frac{n + 1 - \lambda}{1 - \delta}, \quad p > \frac{s(n - 1) - \lambda + 1}{n - s(1 - \delta) - \lambda + 1}.$$

Let G be an s -John domain in \mathbb{R}^n such that $\dim_{\mathcal{M}}(\partial G) \leq \lambda$. Then G supports the fractional $(1, p)$ -Poincaré inequality (1.1).

We need preparations for the proof of Theorem 5.1.

By scaling we may assume that $\text{diam}(G) \leq 1$. Hence, the side lengths of all Whitney cubes in $W = W_G$ are bounded by one and

$$(5.3) \quad W = \bigcup_{j=0}^{\infty} W_j,$$

where each W_j stands for the family of cubes $A \in W$ with $\ell(A) = 2^{-j}$.

For a given s -John domain G , we consider its chain decomposition that is constructed in Section 4. Given $j, k \in \mathbb{N}$ and $\sigma \geq 1$ we define

$$W_{j,k,\sigma} := \{A \in W_j \mid 2^{-(j-k)n} \leq |\cup A(W)| \leq \sigma \cdot 2^{-(j-k-1)n}\}.$$

The following lemma from [HH-SV, Lemma 4.7] gives the properties we need for this chain decomposition of G .

The integer part of $\alpha \in \mathbb{R}$ is denoted by $[\alpha]$.

5.4. Lemma. *Let $s > 1$ and let G be an s -John domain in \mathbb{R}^n such that $\text{diam}(G) \leq 1$ and $\dim_{\mathcal{M}}(\partial G) < \lambda \in [n-1, n)$. Then, there is a constant $\sigma \geq 1$ such that*

$$(5.5) \quad W_j = \bigcup_{k=0}^{[j-j/s]} W_{j,k,\sigma} \quad \text{for every } j \in \mathbb{N}.$$

Furthermore, if $k \in \{0, 1, \dots, [j-j/s]\}$, then

$$(5.6) \quad \#W_{j,k,\sigma} \leq c 2^{-kn} 2^{j(n+1+(\lambda-n-1)/s)}.$$

The positive constant c depends on n , s , ∂G , and the s -John constant of the domain G .

We are ready for the proof of Theorem 5.1.

Proof of Theorem 5.1. Choose $\lambda' \in (\lambda, n)$ such that (5.2) is true if λ is replaced by λ' . Then $\dim_{\mathcal{M}}(\partial G) < \lambda'$ and hence we may assume that $\dim_{\mathcal{M}}(\partial G)$ is strictly less than $\lambda \in [n-1, n)$.

By Theorem 3.1 it is enough to prove the finiteness of

$$\Sigma := \sum_{A \in W} \left(\sum_{Q \in A(W)} |Q| |A|^{\delta/n-1/p} \right)^{p/(p-1)} = \sum_{A \in W} (|\cup A(W)| |A|^{\delta/n-1/p})^{p/(p-1)},$$

where the chain decomposition of G is given by Lemma 5.4. By (5.3) and (5.5) in Lemma 5.4

$$\Sigma = \sum_{j=0}^{\infty} \sum_{k=0}^{[j-j/s]} \sum_{A \in W_{j,k,\sigma}} (|\cup A(W)| |A|^{\delta/n-1/p})^{p/(p-1)}.$$

Then, by using the definition of $W_{j,k,\sigma}$ and (5.6) from Lemma 5.4 we obtain the estimate

$$\begin{aligned}\Sigma &\lesssim \sum_{j=0}^{\infty} \sum_{k=0}^{[j-j/s]} 2^{-kn} 2^{j(n+1+(\lambda-n-1)/s)} \cdot (2^{-(j-k)n} \cdot 2^{-jn(\delta/n-1/p)})^{p/(p-1)} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{[j-j/s]} 2^{kn(p/(p-1)-1)} 2^{j(n+1+(\lambda-n-1)/s-np/(p-1)-\delta p/(p-1)+n/(p-1))}.\end{aligned}$$

Let us fix j and k as in the summation above. Then,

$$kn\left(\frac{p}{p-1} - 1\right) \leq n(j - j/s)\left(\frac{p}{p-1} - 1\right) = \frac{jn(1 - 1/s)}{p-1}.$$

The trivial estimate $[j - j/s] \leq j$ implies that

$$\begin{aligned}\Sigma &\lesssim \sum_{j=0}^{\infty} j \cdot 2^{jn(1-1/s)/(p-1)+n+1+(\lambda-n-1)/s-np/(p-1)-\delta p/(p-1)+n/(p-1)} \\ &= \sum_{j=0}^{\infty} j \cdot 2^{j(ns-s+\lambda p-\lambda-np-p+1-p(\delta-1)s)/s(p-1)}.\end{aligned}$$

By (5.2) the last series converges. \square

6. SHARPNESS OF THEOREM 5.1

We show that Theorem 5.1 is sharp by proving Theorem 6.9. For this purpose we construct s -John domains which do not support the fractional $(1, p)$ -Poincaré inequality (1.1) for certain values of p .

Let us recall the construction of the s -version of a given 1-John domain G , [HH-SV]. We may assume that the diameter of G is restricted by condition

$$(6.1) \quad (\ell(Q)/8)^s \leq \ell(Q)/32, \quad \text{if } Q \in W_G.$$

Let Q be a closed cube in \mathbb{R}^n centered at $x = (x_1, \dots, x_n)$ and whose side length $\ell = \ell(Q)$ satisfies $(\ell/8)^s \leq \ell/32$. Thus, $Q = \prod_{i=1}^n [x_i - \ell/2, x_i + \ell/2]$. The *room* in Q is the open cube

$$R(Q) := \text{int}\left(\frac{1}{4}Q\right) = \prod_{i=1}^n (x_i - \ell/8, x_i + \ell/8)$$

centered at x with side length $\ell/4$. The s -*passage* in Q is the open set

$$P_s(Q) := \left(\prod_{i=1}^{n-1} (x_i - (\ell/8)^s, x_i + (\ell/8)^s) \right) \times (x_n - \ell/8, x_n + \ell/4).$$

Since $(\ell/8)^s < \ell/8$, we have $P_s(Q) \subset \frac{1}{2}Q$. The *long s -passage* in Q is the open set

$$L_s(Q) := \left(\prod_{i=1}^{n-1} (x_i - (\ell/8)^s, x_i + (\ell/8)^s) \right) \times (x_n, x_n + \ell/2) \subset Q.$$

The s -apartment in Q is the set

$$(6.2) \quad A_s(Q) := L_s(Q) \cup (Q \setminus (\partial R(Q) \cup \partial P_s(Q))) \subset Q.$$

6.3. Definition. Let G in \mathbb{R}^n be a 1-John domain and let $s > 1$ be a number such that (6.1) holds. Then, the s -version of G is the domain

$$G_s := Q_0 \cup \bigcup_{\substack{Q \in W_G \\ Q \neq Q_0}} A_s(Q).$$

Here $Q_0 \in W_G$ is the cube containing the 1-John center x_0 of G .

We construct test functions. Let $Q \in W_G$ be fixed, and define the *tiny s -passage* in Q to be the open set

$$T_s(Q) := \left(\prod_{i=1}^{n-1} (x_i - (\ell/8)^s, x_i + (\ell/8)^s) \right) \times (x_n + 5\ell/32, x_n + 7\ell/32).$$

Then, we define a continuous function

$$u^{A_s(Q)} : G_s \rightarrow \mathbb{R}$$

which has linear decay along the n^{th} variable in $T_s(Q)$ and is constant in both components of $P_s(Q) \setminus T_s(Q)$, and satisfies

$$(6.4) \quad u^{A_s(Q)}(x) = \begin{cases} \ell(Q)^{(\lambda-n)/q}, & \text{if } x \in R(Q); \\ 0, & \text{if } x \in G_s \setminus (R(Q) \cup P_s(Q)). \end{cases}$$

In the sense of distributions in G_s ,

$$(6.5) \quad \nabla u^{A_s(Q)} = (0, \dots, 0, -16\ell(Q)^{(\lambda-n)/q-1} \chi_{T_s(Q)})$$

pointwise almost everywhere.

The reason why we do not let $u^{A_s(Q)}$ have linear decay along the whole s -passage $P_s(Q)$ is that we need the following property.

6.6. Remark. Let $Q \in W_G$. Suppose that $x \in G_s$ and $y \in B^n(x, \text{dist}(x, \partial G_s))$ are such that

$$|u^{A_s(Q)}(x) - u^{A_s(Q)}(y)| \neq 0.$$

Then x and y both belong to $P_s(Q)$. This fact follows from the assumption (6.1).

The following proposition is the main tool for proving Theorem 6.9.

6.7. Proposition. Let G be a 1-John domain in \mathbb{R}^n and $s > 1$ be such that (6.1) holds. Suppose that

$$\limsup_{k \rightarrow \infty} 2^{-\lambda k} \cdot \#W_k > 0, \quad \text{where } \lambda = \dim_{\mathcal{M}}(\partial G) \in [n-1, n).$$

Let $\delta, \tau \in (0, 1)$ and $1 \leq q < p < \infty$ be such that

$$(6.8) \quad \frac{(p-q)(\lambda-n)}{pq} + \frac{(s-1)(n-1)}{p} \geq 1 - s(1-\delta).$$

Then the s -version of G is an s -John domain with $\dim_{\mathcal{M}}(\partial G_s) = \lambda$ and G_s does not support the fractional (q, p) -Poincaré inequality (1.1).

Proof. The fact

$$\dim_{\mathcal{M}}(\partial G_s) = \dim_{\mathcal{M}}(\partial G) = \lambda$$

is from [HH-SV, Proposition 5.11]. By [HH-SV, Proposition 5.16], the domain G_s is an s -John domain. Hence, it remains to prove the failure of the fractional Poincaré inequality.

Let us choose $k_0 \in \mathbb{N}$ such that

$$\limsup_{k \rightarrow \infty} 2^{-\lambda(k-k_0)} \cdot \#W_k > 2.$$

This allows us to choose indices $j(k)$, $k \in \mathbb{N}$, inductively such that

$$\max\{k_0, -\log_2 \ell(Q_0)\} < j(1) < j(2) < \dots$$

and $\#W_{j(k)} \geq 2 \cdot 2^{\lambda(j(k)-k_0)}$ for every $k \in \mathbb{N}$. Let us write $M_j := 2^{\lfloor \lambda(j-k_0) \rfloor}$, where $\lfloor \lambda(j-k_0) \rfloor$ means the integer part of $\lambda(j-k_0)$, and let us choose cubes

$$Q_{j(k)}^1, \dots, Q_{j(k)}^{2M_{j(k)}} \in W_{j(k)} \setminus \{Q_0\}.$$

For every $m \in \mathbb{N}$ we define

$$v_m := \sum_{k=1}^m \left(\sum_{i=1}^{M_{j(k)}} u^{A_s(Q_{j(k)}^i)} - \sum_{i=M_{j(k)}+1}^{2M_{j(k)}} u^{A_s(Q_{j(k)}^i)} \right).$$

Note that $(v_m)_{G_s} = 0$ and

$$\begin{aligned} A_m &:= \left(\int_{G_s} |v_m - (v_m)_{G_s}|^q \right)^{1/q} = \left(\sum_{k=1}^m \sum_{i=1}^{2M_{j(k)}} \int_{G_s} |u^{A_s(Q_{j(k)}^i)}|^q \right)^{1/q} \\ &\geq \left(m \cdot 2 \cdot 2^{\lambda(j(k)-k_0)-1} \cdot 2^{-j(k)(\lambda-n)} \cdot 4^{-n} \cdot 2^{-j(k)n} \right)^{1/q} = c_{n,q,\lambda,k_0} m^{1/q}. \end{aligned}$$

Next we estimate the right hand side of (1.1) with $u = v_m$. We write

$$G_s(x) := B^n(x, \text{dist}(x, \partial G_s)) \subset G_s \quad \text{for } x \in G_s.$$

Remark 6.6 yields: if $x \in G_s$ and $y \in G_s(x)$ are such that $|v_m(x) - v_m(y)| \neq 0$, then $x, y \in P_s(Q)$ for some Whitney cube $Q \in W_G$. By using this we obtain

$$\begin{aligned} B_m &:= \left(\int_{G_s} \int_{G_s(x)} \frac{|v_m(x) - v_m(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p} \\ &= \left(\sum_{Q \in W_G} \int_{Q \cap G_s} \int_{G_s(x)} \frac{|v_m(x) - v_m(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p} \\ &= \left(\sum_{Q \in W_G} \int_{P_s(Q)} \int_{P_s(Q) \cap G_s(x)} \frac{|v_m(x) - v_m(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p} \\ &= \left(\sum_{k=1}^m \sum_{i=1}^{2M_{j(k)}} \int_{P_s(Q_{j(k)}^i)} \int_{P_s(Q_{j(k)}^i) \cap G_s(x)} \frac{|u^{A_s(Q_{j(k)}^i)}(x) - u^{A_s(Q_{j(k)}^i)}(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p}. \end{aligned}$$

Let us fix a cube $R = Q_{j(k)}^i$, where $k \in \{1, \dots, m\}$ and $i \in \{1, 2, \dots, 2M_{j(k)}\}$. By (6.5)

$$|u^{A_s(R)}(x) - u^{A_s(R)}(y)| \leq 16\ell(R)^{(\lambda-n)/q-1}|x-y|, \quad x, y \in P_s(R).$$

Hence,

$$\begin{aligned} \mathcal{I}_R &:= \int_{P_s(R)} \int_{P_s(R) \cap G_s(x)} \frac{|u^{A_s(R)}(x) - u^{A_s(R)}(y)|^p}{|x-y|^{n+\delta p}} dy dx \\ &\lesssim \ell(R)^{p(\lambda-n)/q-p} \int_{P_s(R)} \int_{P_s(R) \cap G_s(x)} |x-y|^{-n+(1-\delta)p} dy dx. \end{aligned}$$

Note that $G_s(x) \subset B^n(x, \ell(R)^s)$ if $x \in P_s(R)$. Thus,

$$\int_{P_s(R) \cap G_s(x)} |x-y|^{-n+(1-\delta)p} dy \leq \int_{B^n(0, \ell(R)^s)} |y|^{-n+(1-\delta)p} dy \lesssim \ell(R)^{s(1-\delta)p},$$

and it follows that

$$\begin{aligned} \mathcal{I}_R &\lesssim \ell(R)^{p(\lambda-n)/q-p} |P_s(R)| \ell(R)^{s(1-\delta)p} \\ &= \ell(R)^{p(\lambda-n)/q-p+s(n-1)+1+s(1-\delta)p} = 2^{-j(k)(p(\lambda-n)/q-p+s(n-1)+1+s(1-\delta)p)}. \end{aligned}$$

These estimates and inequality (6.8) yield

$$B_m \lesssim \left(\sum_{k=1}^m 2^{\lambda j(k)} 2^{-j(k)(p(\lambda-n)/q-p+s(n-1)+1+s(1-\delta)p)} \right)^{1/p} \lesssim m^{1/p}.$$

By using the assumption $q < p$ we obtain

$$\frac{A_m}{B_m} \geq c_{n,s,p,q,k_0,\lambda,\delta} m^{1/q-1/p} \xrightarrow{m \rightarrow \infty} \infty.$$

Hence, the domain G_s does not support the fractional (q, p) -Poincaré inequality (1.1) for any $\tau \in (0, 1)$. \square

The following theorem shows the sharpness of Theorem 5.1.

6.9. Theorem. *Let $s > 1$, $p \in (1, \infty)$, $\lambda \in [n-1, n)$, and let $\delta, \tau \in (0, 1)$. Suppose that*

$$s < \frac{n+1-\lambda}{1-\delta}, \quad p \leq \frac{s(n-1)-\lambda+1}{n-s(1-\delta)-\lambda+1}.$$

Then, there is an s -John domain G_s in \mathbb{R}^n with the following properties: $\dim_{\mathcal{M}}(\partial G_s) = \lambda$ and G_s does not support the fractional $(1, p)$ -Poincaré inequality (1.1).

Proof. By [HH-SV, Proposition 5.2] there is a 1-John domain G in \mathbb{R}^n such that $\dim_{\mathcal{M}}(\partial G) = \lambda$ and $\limsup_{k \rightarrow \infty} 2^{-\lambda k} \cdot \#W_k > 0$. By scaling we may also assume that (6.1) holds. Hence, by Proposition 6.7 the s -version G_s has required properties. \square

REFERENCES

- [A] Robert A. Adams, *Sobolev spaces*, Academic Press, Inc., Orlando, Florida, 1975.
- [B] Bogdan Bojarski, *Remarks on Sobolev imbedding inequalities*, Complex Analysis Joensuu 1987, Lecture Notes in Math., vol. 1351, Springer, 1988, 52–68.
- [BBM1] Jean Bourgain, Haïm Brezis and Petru Mironescu, *Another look at Sobolev spaces*, Optimal Control and Partial Differential Equations, edited by J. L. Menaldi, E. Rofman, and A. Sulem, IOS Press, 439–455 (2001).
- [BBM2] Jean Bourgain, Haïm Brezis and Petru Mironescu, *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications*, J. Anal. Math., **87** (2002), 77–101.
- [GT] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [HK] Piotr Hajláz and Pekka Koskela, *Sobolev met Poincaré*, Memoirs Amer. Math. Soc., **688** (2000), 1–101.
- [HH-SV] Petteri Harjulehto, Ritva Hurri-Syrjänen and Antti V. Vähäkangas, *On the $(1, p)$ -Poincaré inequality*, University of Helsinki, Department of Mathematics and Statistics Report series, **519** (2011).
- [He] Lars Inge Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc., **36** (1972), 505–510.
- [Hu] Ritva Hurri, *Poincaré domains in \mathbb{R}^n* , Ann. Acad. Sci. Fenn. Ser. A Math. Dissertationes, **71** (1988).
- [IV] Lizaveta Ihnatsyeva and Antti V. Vähäkangas, *Characterization of traces of smooth functions on Ahlfors regular sets*, arXiv:1109.2248 (2011).
- [MS1] Vladimir Maz'ya and Tatyana Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal., **195** (2002), 230–238.
- [MS2] Vladimir Maz'ya and Tatyana Shaposhnikova, *Erratum to “On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces”*, J. Funct. Anal., **201** (2003), 298–300.
- [N] Tomi Nieminen, *Generalized mean porosity and dimension*, Ann. Acad. Sci. Fenn. Math. Ser. A I Math., **31** (2006), 143–172.
- [P1] Augusto C. Ponce, *A variant of Poincaré inequality*, C. R. Acad. Sci. Paris Ser. I, **337** (2003), 253–257.
- [P2] Augusto C. Ponce, *An estimate in the spirit of Poincaré's inequality*, J. Eur. Math. Soc., **6** (2004), 1–15.
- [S] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.

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